On the quadratic Morse-Smale endomorphisms

To cite this article: Svetlana S. Belmesova 2018 J. Phys.: Conf. Ser. 990 012001

View the article online for updates and enhancements.

Related content
- Continuous Morse-Smale flows with three equilibrium positions
  E V Zhuzhoma and V S Medvedev
- SERIES IN THE SYSTEM
  \( \{f(nx)\}_{n=1}^{\infty} \)
  A V Kasyanchuk
- RADICALS OF ENDMORPHISM RINGS
  OF TORSION-FREE ABELIAN GROUPS
  P A Krylov
On the quadratic Morse-Smale endomorphisms

Svetlana S. Belmesova
National Research Nizhni Novgorod State University
Nizhny Novgorod, Russia
E-mail: belmesovass@mail.ru

Abstract. For the quadratic map $F\mu = (xy, (x - \mu)^2)$ we indicate an interval of parameter values, such that map $F\mu$ with a parameter value in the indicated interval is a (nonsingular) Morse-Smale endomorphism.

1. Introduction

The bibliography of investigation of polynomial maps, and in particular, of quadratic maps is very extensive (see, for example, [1] – [10]). In this paper we consider the quadratic maps on the plane from the one-parameter family of the type (1)

$$F\mu(x, y) = (xy, (x - \mu)^2),$$

(1)

where $(x, y)$ is an arbitrary point of the plane $xOy$ and $\mu \in (0, 1]$. Different aspects of dynamics of the maps from one-parameter family (1) are investigated in [7] – [10]. This paper is a continuation of [7] – [10]. We prove here, that the map $F\mu$ is a (nonsingular) Morse-Smale endomorphism for any $\mu \in (0, 1)$ and the map $F_1$ is a singular Morse-Smale endomorphism.

We will extend the definition of a (nonsingular) Morse-Smale endomorphism acting on a compact manifold [11] to the case of maps of a plane.

Nonsingular Morse-Smale flows were considered in [12]. Let us define a nonsingular Morse-Smale endomorphism by analogy with [12].

In the case of endomorphisms, the global stable manifold of a point is not necessarily connected. Therefore, in Definition 1 we use the local stable manifold $W^s_{loc}(p)$ of a $p$; we understand it as a connected component of the global stable manifold $W^s_p$ containing the point $p$ [13].

Definition 1. An endomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ is called a (nonsingular) Morse-Smale endomorphism, if

(i) the nonwandering set $\Omega(F)$ is finite and consists of hyperbolic periodic points;
(ii) the local stable manifold $W^s_{loc}(p)$ and the global unstable manifold $W^u(q)$ of different periodic points $p$, $q$ intersect transversally, i.e., if $(x, y) \in W^s_{loc}(p) \cap W^u(q)$, then $T_{(x,y)}W^u_{loc}(p) \oplus T_{(x,y)}W^u(q)$ holds.

Definition 2. An endomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ with a finite nonwandering set $\Omega(F)$ is called a singular Morse-Smale endomorphism, if at least one of the following conditions is fulfilled:
There are three fixed points \( J \) consists of two fixed points (the nonhyperbolic point \( F \) coordinate functions of \( F \)). In this part of the paper we give the preliminary information that will be used.

2. The main technical results

In Section 3 we give the description of nonwandering set \( \Omega(\mu) \) of periodic two, which is formed by two sources \( B \) and \( \) and complete the proof of Theorem 1.

Theorem 1. Let \( F^\mu \) be a map of type (1). Then for each \( \mu \in (0,1) \) the map \( F^\mu \) is a (nonsingular) Morse-Smale endomorphism, such that its nonwandering set \( \Omega(F^\mu) \) consists of three fixed points (the sink \( A_1(0; \mu^2) \), the source \( A_2(\mu+1; 1) \) and the saddle point \( A_3(\mu-1; 1) \)) and periodic orbit \( B \) of period two, which is formed by two sources \( B_1(2-\sqrt{2}; \frac{1}{3+2\sqrt{2}}) \), \( B_2(2+\sqrt{2}; \frac{1}{3+2\sqrt{2}}) \).

This paper is organized as follows. Section 1 contains a main technical results. In Section 2 we construct the special partition of the first quadrant, which we use for the proof of Theorem 1. In Section 3 we give the description of nonwandering set \( \Omega(F^\mu) \) of the map \( F^\mu \) and complete the proof of Theorem 1.

2. The main technical results

In this part of the paper we give the preliminary information that will be used.

We begin this section with a formulation of the main properties of map \( F^\mu \).

Let us \( F^\mu_n \) is the \( n \)-th iteration of map (1) and \( f_{\mu,n} \) and \( y_{\mu,n} \) are the first and the second coordinate functions of \( F^\mu_n \) respectively, i.e. \( F^\mu_n(x,y) = (f_{\mu,n}(x,y), y_{\mu,n}(x,y)) \) for any \( n \geq 1 \).

Denote by \( J(F^\mu_n(x,y)) = \frac{\partial F^\mu_n(x,y)}{\partial (x,y)} \) and \( J(F^\mu_n(x,y)) \) Jacobian matrix and Jacobian of \( F^\mu_n \) respectively.

Proposition 1. Map (1) for every \( \mu \in (0,1] \) possesses the properties:

(i) The equality \( J(F^\mu_n(x,y)) = 0 \) \( (n \geq 1) \) holds off coordinates of a point \( (x; y) \) satisfy one of the equations \( f_{\mu,i}(x,y) = 0 \) or \( f_{\mu,i}(x,y) = \mu \), for some \( 0 \leq i \leq n-1 \); in particular, the critical set of the map \( F^\mu_n \) consists of two straight lines \( C_0 : x = 0 \) and \( C_\mu : x = \mu \).

(ii) The following inclusions hold for every open quadrant \( K_i \) \( (i = 1, 2, 3, 4) \) of the plane \( xOy \): \( F^\mu_n(K_3) \subset K_1, K_4 \subset K_2, F^\mu_n(K_2) \subset K_1, K_4 \subset K_1 \), where \( (\cdot) \) means the closure of a set.

(iii) Triangle \( \Delta_\mu = \{(x,y) : x, y \geq 0, \frac{x^2}{\mu^2} + \frac{y^2}{\mu^2} \leq 1 \} \) is \( F^\mu \) - invariant set 1 such that its legs \( k_{\mu,x} = \{(x,0) : 0 \leq x \leq 2\mu \} \) and \( k_{\mu,y} = \{(0,y) : 0 \leq y \leq \mu^2 \} \) satisfy the equalities \( F^\mu_{\mu}(k_{\mu,x}) = k_{\mu,y} \), \( F^\mu_{\mu}(k_{\mu,y}) = (0,\mu^2) \); in addition the restriction \( F^\mu_{\mu} \) on its hypotenuse \( h_\mu \) is defined by the equality \( F^\mu_{\mu}(h_\mu(x,y)) = (x\mu(\mu - \frac{\mu^2}{2}); (\mu - \frac{\mu^2}{2}) \) such that the image \( F^\mu_{\mu}(h_\mu(x,y)) \) of hypotenuse \( h_\mu \) is the closed interval of the line \( 2x + \mu y = \mu^3 \) which lies in the triangle \( \Delta_\mu \).

(iv) There are three fixed points \( A_1(0; \mu^2) \), \( A_2(\mu+1; 1) \), \( A_3(\mu-1; 1) \), such that \( A_1(0; \mu^2) \) is a sink for any \( \mu \in (0,1) \) and is a non-hyperbolic point when \( \mu = 1 \), \( A_2(\mu+1; 1) \) is a source for any \( \mu \in (0,1] \) and point \( A_3(\mu-1; 1) \) is a saddle point for any \( \mu \in (0,1] \).

1 We say that set \( A, A \subset R^2 \) is \( F^\mu \) - invariant set if the inclusion \( F^\mu(A) \subset A \) holds.
(v) There exists a unique periodic orbit $B$ with the least period two, which formed by two sources
$$B_1 \left( \frac{\sqrt{2+\mu}}{\mu} : x \geq \frac{\sqrt{2+\mu}}{\mu} \right) \text{ and } B_2 \left( \frac{\sqrt{2+\mu}}{\mu} : x \leq \frac{\sqrt{2+\mu}}{\mu} \right).$$

(vi) The unbounded set $D_{\mu, +\infty} = \{(x, y) \in K_1 : x \geq \mu + 1, y \geq 1\}$ and the closed set $D_{\mu, A_1} = \{(x, y) \in K_1 : 0 < x \leq \mu + 1, 0 < y \leq 1\}$ are $F_\mu$-invariant sets; for every point $(x, y) \in D_{\mu, +\infty} \setminus \{A_2(\mu + 1; 1)\}$ the following equalities hold:
$$\lim_{n \to +\infty} f_{\mu, n}(x, y) = \lim_{n \to +\infty} g_{\mu, n}(x, y) = +\infty;$$
and for any point $(x, y) \in D_{\mu, A_1} \setminus \{A_2(\mu + 1; 1)\}$ the following equalities hold:
$$\lim_{n \to +\infty} f_{\mu, n}(x, y) = 0, \quad \lim_{n \to +\infty} g_{\mu, n}(x, y) = \mu^2;$$
in addition, the restriction $F_\mu|_{D_{\mu, +\infty}}$ is diffeomorphism of set $D_{\mu, +\infty}$ on its image $F_\mu(D_{\mu, +\infty})$ such that $F_\mu|_{D_{\mu, +\infty}} \subset D_{\mu, +\infty}.$

To prove Theorem 1 we need also the following property of the map $F_\mu.$

**Proposition 2** [8]. For any $\mu \in (0, 1]$ there exists a $C^1$-smooth strictly decreasing function $y = \Gamma_\mu(x)$ defined on the interval $(\mu, +\infty)$, such that $\Gamma_\mu((\mu, +\infty)) = (0, +\infty)$; moreover, the graph $\Gamma_\mu$ of this function is a $F_\mu$-invariant curve belonging to the unbounded noninvariant set $\{A_2(\mu + 1; 1)\} \cup (K_1 \setminus (D_{\mu, +\infty} \cup D_{\mu, A_1})).$

The invariant curve $\Gamma_\mu$ makes it possible to construct the partition of the first quadrant and to prove that $F_\mu$ is a (nonsingular) Morse-Smale endomorphism for every $\mu \in (0, 1)$ and the map $F_1$ is a singular Morse-Smale endomorphism.

Note, that the source $A_2(\mu + 1; 1)$ is the limit point of the preimages of unbounded rank $\{F_\mu^{-1}(C_\mu \cap K_1)\}_{i \geq 1}$ of the part of critical line $C_\mu$ in the first open quadrant $K_1$ and $\{F_\mu^{-1}(C_0 \cap K_1)\}_{i \geq 1}$ are preimages of unbounded rank of the part of critical line $C_0$ in $K_1$ ($\{F_\mu^{-1}(C_\mu \cap K_1)\}_{i \geq 1} \subseteq \{F_\mu^{-1}(C_0 \cap K_1)\}_{i \geq 1}$ since $F_\mu^2(C_\mu) = C_0$ for $\mu \in (0, 1]$. It means, that there not exist universal neighborhood $U(A_2)$ of the source $A_2$, such that the restriction $F_\mu|_{U(A_2)}$ is diffeomorphism for all $n \geq 1.$

Calculating eigenvectors of the differential $DF_\mu(A_2)$ and using claim 6 of Proposition 1, we obtain, that one of unit eigenvectors $p_{A_2} = \left( \frac{1 - \sqrt{3+8\mu}}{2\sqrt{\sqrt{2+8\mu}+2\sqrt{9+8\mu}}}, \frac{4}{\sqrt{\sqrt{2+8\mu}+2\sqrt{9+8\mu}}} \right)$ of $DF_\mu(A_2)$ of the map $F_\mu$ lies in the unbounded invariant set $D_{\mu, +\infty}$ and the other unit eigenvector $q_{A_2} = \left( \frac{1 - \sqrt{3+8\mu}}{2\sqrt{\sqrt{2+8\mu}+2\sqrt{9+8\mu}}}, \frac{4}{\sqrt{\sqrt{2+8\mu}+2\sqrt{9+8\mu}}} \right)$ lies in the unbounded noninvariant set $K_1 \setminus D_{\mu, +\infty}.$

Therefore, the standard technique of cones and normal forms can not be applied for the proof of existence of the invariant curve $\Gamma_\mu.$

For the proof of the Proposition 2 we use the nonlocal theorem of existence of unlocal strictly decreasing implicit functions $y = \eta_{\mu, n}(x), n \geq 1$ [8]. The function $y = \eta_{\mu, n}(x)$ is the solution of equation $f_{\mu, n}(x, y) = \mu + 1$ on the interval $(\mu, +\infty)$ for any $n \geq 2$ and the function $y = \eta_{\mu, 1}(x)$ is the solution of equation $f_{\mu, 1}(x, y) = \mu + 1$ on the interval $(0, +\infty)$. The graph $\eta_{\mu, n}$ of this function for all $n \geq 1$ contains the unique fixed point – source $A_2(\mu + 1; 1)$ and does not contain the common points with the periodic orbit $B$ of the period two. The properties of the function $y = \eta_{\mu, n}(x), n \geq 1$ are formulated below, in Proposition 3.

**Proposition 3.** Let $F_\mu$ be the map $(1), \mu \in (0, 1]$. Then:

(i) for any $n \geq 1$ the following relations hold:

(a) $\eta_{\mu, 2n-1} < \eta_{\mu, 2n+1}, \eta_{\mu, 2n+2} < \eta_{\mu, 2n}$ for any $x \in (\mu, \mu + 1)^2.$

\[\] We say that $\eta_{\mu, k}$ is preceded $\eta_{\mu, l}$ for $k \neq l, k, l \geq 1$ and denote by $\eta_{\mu, k} < \eta_{\mu, l}$ if the inequality $\eta_{\mu, k}(x) < \eta_{\mu, l}(x)$ holds for any $x \in (\mu, \mu + 1)$ (for any $x \in (\mu + 1, +\infty)).$
(b) \( \eta_{2n} < \eta_{2n+2}, \eta_{2n+1} < \eta_{2n} \) for any \( x \in (\mu + 1, +\infty) \).

(ii) the sequence of the restrictions \( \{ \eta_{2n}|_{[a, \mu+1]}(x) \}_{n \geq 1} \) of \( C^1 \) smooth strictly decreasing functions \( y = \eta_{2n}(x) \) (\( y = \eta_{2n+1}(x) \)), \( n \geq 1 \) on the interval \([a, \mu+1]\) (resp. \([\mu+1, b]\)) to the strictly decreasing function \( y = \Gamma_\mu(x) \) restricted on the interval \([a, \mu+1]\) (resp. \([\mu+1, b]\)).

Note also, that the preimages of the different orders of the straight line \( x = \mu + 1 \) and the critical line \( C_\mu \) intersect each other under curve \( \Gamma_\mu \). Therefore, for the proof Theorem 1 we need the sets:

\[
\begin{align*}
D^g_{\mu} &= \{(x; y) : x \in (\mu, +\infty), y > \Gamma_\mu\}; \\
D^f_{\mu} &= \{(x; y) : x \in (0, +\infty), 0 < y < \Gamma_\mu\} \quad \text{(see Fig. 1)}.
\end{align*}
\]

**Figure 1.** Intersections of preimages of the straight line \( x = \mu + 1 \) with the preimages of the critical line \( C_\mu \) in double and triple points, which lie in the \( K_1 \).

The set of intersection points of the above preimages is described in the following lemma.

**Lemma 1.** Let \( F_\mu \) be the map (1), \( \mu \in (0, 1] \). Then

(i) for any odd number \( n \geq 1 \) and for any \( x \in (0, \mu+1) \) the curve \( f_{\mu,n}(x,y) = \mu + 1 \) has in \( D^g_{\mu} \) at least one triple point \( M^{(3)}_{n} \) (i.e. a point of the intersection of three curves \( f_{\mu,n}(x,y) = \mu + 1, f_{\mu,n+1}(x,y) = \mu \) and \( f_{\mu,n+2}(x,y) = \mu \)) at least one double point \( M^{(2,1)}_{n} \) of the first type (i.e. a point of the intersection of two curves \( f_{\mu,n}(x,y) = \mu + 1 \) and \( f_{\mu,n-1}(x,y) = \mu \)) and a countable set of double points \( M^{(2,2)}_{n,j} \) of the second type (i.e. the points of intersection of the curve \( f_{\mu,n}(x,y) = \mu + 1 \) with every curve \( f_{\mu,j}(x,y) = \mu \) for any \( j \geq n + 3, n \geq 1 \));
(ii) for any even number \( n \geq 1 \) and for any \( x \in (\mu + 1, +\infty) \) the curve \( f_{\mu,n}(x, y) = \mu + 1 \) has in \( D^a_{\mu} \) at least one triple point \( M_{n}^{(3)} \), at least one double point \( M_{n}^{(2,1)} \) of the first type and countable set of double points \( M_{n,j}^{(2,2)} \) of the second type;

(iii) the set \( D^a_{\mu} \) doesn’t contain points of intersection of the curves \( f_{\mu,n}(x, y) = \mu + 1 \) and \( f_{\mu,m}(x, y) = \mu \) \((m, n \geq 1)\) other than above.

3. The special partition of the first quadrant \( K_1 \) on the plane

In this section we construct the partition in \( K_1 \) using the properties of the invariant curve \( \Gamma_{\mu} \), the properties of strictly decreasing functions \( y = \eta_{\mu,n}(x) \) and the preimages of the different orders of the critical line \( C_{\mu} \). This partition help us to investigate the asymptotic behaviour of \( F_{\mu} \) – trajectories in the sets \( D^a_{\mu} \) and \( D^b_{\mu} \).

**Proposition 4.** Let \( F_{\mu} \) be a map of type \((1)\), \( \mu \in (0, 1] \). Then

(i) the set \( D^a_{\mu} \) is invariant set and for every point \((x, y) \in D^a_{\mu}\) the equalities hold

\[
\lim_{n \to +\infty} f_{\mu,n}(x, y) = +\infty, \quad \lim_{n \to +\infty} g_{\mu,n}(x, y) = +\infty.
\]

(ii) the set \( D^b_{\mu} \) is invariant set and for every point \((x, y) \in D^b_{\mu}\) with expection of the periodic orbit \( B \) of period two and all of its preimages the equalities hold

\[
\lim_{n \to +\infty} f_{\mu,n}(x, y) = 0, \quad \lim_{n \to +\infty} g_{\mu,n}(x, y) = \mu^2.
\]

We divide the proof of Proposition 4 in some steps.

**I.** Let us consider the set \( D^a_{\mu} \). For the proof of the claim 1 of Proposition 4 we define the sets:

\[
T_{D^a_{\mu},2n} = \{(x; y) \in D^a_{\mu} : \eta_{\mu,2n+2|\mu+1}(x) \leq y \leq \eta_{\mu,2n+2|\mu+1}(x)\} \quad \text{where} \quad a \in (\mu, \mu + 1);
\]

\[
T_{D^a_{\mu},2n-1} = \{(x; y) \in D^a_{\mu} : \eta_{\mu,2n+1|\mu+1}(x) \leq y \leq \eta_{\mu,2n+1|\mu+1}(x)\} \quad \text{where} \quad b \in (\mu + 1, +\infty);
\]

\[
T_{D^a_{\mu},+\infty} = \{(x; y) \in D^a_{\mu} : y \geq \eta_{\mu,2}\}; \quad T_{D^a_{\mu},+\infty} = \{(x; y) \in D^a_{\mu} : y \geq \eta_{\mu,1}\}.
\]

Then from Proposition 3 we obtain the following equality

\[
D^a_{\mu} = D_{\mu, +\infty} \cup T_{D^a_{\mu},+\infty} \cup \bigcup_{n \geq 1} T_{D^a_{\mu},2n} \cup \bigcup_{n \geq 1} T_{D^a_{\mu},2n-1},
\]

such that the inclusions are valid

\[
F_{\mu}(T^l_{D^a_{\mu},2n}) \subset T^l_{D^a_{\mu},+\infty}, \quad F_{\mu}(T^r_{D^a_{\mu},+\infty}) \subset T^r_{D^a_{\mu},+\infty} \quad \text{and} \quad F_{\mu}(T^r_{D^a_{\mu},+\infty}) \subset D_{\mu, +\infty}.
\]

**Lemma 2.** For all \( \mu \in (0, 1] \) and any \( n \geq 1 \) the map \( F_{\mu} \) is the \( C^1 \) - diffeomorphism of the set \( T_{D^a_{\mu},2n} \) on the set \( T_{D^a_{\mu},2n-1} \).

**Proof.** Indeed, by Proposition 3 and Lemma 1 the restriction \( F_{\mu}|_{T^l_{D^a_{\mu},2n}} \) is surjective local diffeomorphism of the set \( T_{D^a_{\mu},2n} \) on the set \( T_{D^a_{\mu},2n-1} \). We show, that \( F_{\mu} : T_{D^a_{\mu},2n} \to T_{D^a_{\mu},2n-1} \) is injective map. Let us \((x_1; y_1)\) and \((x_2; y_2)\) are an arbitrary points in \( T_{D^a_{\mu},2n} \), such that \((x_1; y_1) \neq (x_2; y_2)\). The coordinate function \( f_{\mu,1}(x, y) = xy \) is the strictly monotone function for each variable and the coordinate function \( g_{\mu,1}(x, y) = (x-\mu)^2 \) is the strictly monotone function for variable \( x \) \((x > \mu)\). We have

\[
(f_{\mu,1}(x_1, y_1), g_{\mu,1}(x_1, y_1)) \neq (f_{\mu,1}(x_2, y_2), g_{\mu,1}(x_2, y_2)).
\]
Then $F_\mu : \Gamma_{\mu,2n}^\alpha \to \Gamma_{\mu,2n-1}^\alpha$ is bijection and there exists the inverse map $(F_\mu|_{\Gamma_{\mu,2n-1}^\alpha})^{-1}$ to the map $F_\mu|_{\Gamma_{\mu,2n}^\alpha}$ with the Jacobian matrix

$$J((F_\mu|_{\Gamma_{\mu,2n}^\alpha})^{-1}(x,y)) = \left( \begin{array}{cc} 0 & \frac{1}{\mu+\sqrt{\beta}} \\ \frac{1}{\mu+\sqrt{\beta}} & \frac{1}{\mu+\sqrt{\beta}} \end{array} \right) |_{\Gamma_{\mu,2n-1}^\alpha}.$$ 

The partial derivatives of coordinate functions of map $(F_\mu|_{\Gamma_{\mu,2n-1}^\alpha})^{-1}$ are continuous on the set $T_{\mu,2n-1}^\alpha$. Therefore, $F_\mu : \Gamma_{\mu,2n}^\alpha \to T_{\mu,2n-1}^\alpha$ is $C^1$ – diffeomorphism. Lemma 2 is proved.

Using Lemma 2, formula (2), inclusions (6) and claim 6 of Proposition 1 we obtain the following result.

**Lemma 3.** For any $\mu \in (0,1]$ set $D_{\mu}^\alpha$ is $F_\mu$ – invariant set and for every point $(x,y) \in D_{\mu}^\alpha$, the following equalities hold:

$$\lim_{n \to +\infty} f_{\mu,n}(x,y) = +\infty, \quad \lim_{n \to +\infty} g_{\mu,n}(x,y) = +\infty.$$ 

Thus, the claim 1 of Proposition 4 is proved.

**II.** Our next step is to prove, that the set $D_{\mu}^\alpha$ is invariant set, such that the trajectories of the points under $G_{\mu}$ satisfy equalities

$$\lim_{n \to +\infty} f_{\mu,n}(x,y) = 0, \quad \lim_{n \to +\infty} g_{\mu,n}(x,y) = \mu^2.$$ 

**II (a).** We begin from the construction the partition of the set $D_{\mu}^\alpha$. For this goal we need the sets:

$T_{\mu,2n-1}^\alpha = \{(x,y) \in D_{\mu}^\alpha : \eta_{\mu,2n-1}|_{[a,b]}(x) \leq y \leq \eta_{\mu,2n+1}|_{[a,b]}(x)\},$ where $[a,b] \subset (\mu,\mu+1)$, $n \geq 1$;

$T_{\mu,2n}^\alpha = \{(x,y) \in D_{\mu}^\alpha : \eta_{\mu,2n}|_{[a,b]}(x) \leq y \leq \eta_{\mu,2n+2}|_{[a,b]}(x)\},$ where $[a,b] \subset (\mu+1, +\infty)$, $n \geq 1$;

$T_{\mu,A_1}^\alpha = \{(x,y) \in K_1 : 0 < x < \mu + 1, 1 < y \leq \frac{\mu+1}{x}\};$

$T_{\mu,A_1}^\alpha = \{(x,y) \in K_1 : \mu + 1, 0 < y \leq \frac{\mu+1}{x(x-\mu)^2}\}.$

**Lemma 4.** For any $\mu \in (0,1]$ the inclusions

$$F_\mu(T_{\mu,A_1}^\alpha) \subset D_{\mu,A_1}, F_\mu(T_{\mu,A_1}^\alpha) \subset T_{\mu,A_1}^\alpha$$

are valid.

**Proof.** We prove the first of the inclusions. Note that, the critical line $C_\mu$ intersect the set $T_{\mu,A_1}^\alpha$. Therefore, we represent set $T_{\mu,A_1}^\alpha$ as the union of two sets:

$$T_{\mu,A_1}^{1,1} = \{(x,y) \in T_{\mu,A_1}^\alpha : x \in (0,\mu)\};$$

$$T_{\mu,A_1}^{1,2} = \{(x,y) \in T_{\mu,A_1}^\alpha : x \in (\mu, \mu+1)\};$$

i. e. $T_{\mu,A_1}^\alpha = T_{\mu,A_1}^{1,1} \cup T_{\mu,A_1}^{1,2}.$
We show, that the following inclusions $F_\mu(T_{D_{\mu,A_1}}^{1}) \subset D_{\mu,A_1}$; $F_\mu(T_{D_{\mu,A_1}}^{2}) \subset D_{\mu,A_1}$ are valid. Let $(x;y)$ is an arbitrary point of the set $T_{D_{\mu,A_1}}^{1}$. By definition of the set $T_{D_{\mu,A_1}}^{1}$ the following inequalities hold: $0 < xy < \mu + 1$, $(x - \mu)^2 < 1$. Hence, $F_\mu(T_{D_{\mu,A_1}}^{1}) \subset D_{\mu,A_1}$.

If $(x;y) \in T_{D_{\mu,A_1}}^{2}$ then $\mu < xy < \mu + 1$, $(x - \mu)^2 < 1$ and the inclusion $F_\mu(T_{D_{\mu,A_1}}^{2}) \subset D_{\mu,A_1}$ is valid. Therefore, $F_\mu(T_{D_{\mu,A_1}}) \subset D_{\mu,A_1}$. Analogously above we prove, that $F_\mu(T_{D_{\mu,A_1}}^{1}) \subset D_{\mu,A_1}$.

Lemma 4 is proved.

Using Lemma 4 and claim 6 of the Proposition 1 we obtain the following result.

**Lemma 5.** The set $D_{\mu,A_1} \cup T_{D_{\mu,A_1}}^{1} \cup T_{D_{\mu,A_1}}^{2}$ is $F_\mu$ - invariant set and for every point $(x;y) \in D_{\mu,A_1} \cup T_{D_{\mu,A_1}}^{1} \cup T_{D_{\mu,A_1}}^{2} \setminus \{A_2(\mu + 1; 1)\}$ the following equalities hold:

$$
\lim_{n \to +\infty} f_{\mu,n}(x, y) = 0, \quad \lim_{n \to +\infty} g_{\mu,n}(x, y) = \mu^2.
$$

**II (b).** In this part of the work we investigate the properties of the set

$G = \{(x;y) \in K_1 : x \in (0, \mu + 1), \eta_{\mu,1}(x) < y < \Gamma_\mu \} \cup \{(x;y) \in K_1 : x > \mu + 1, \eta_{\mu,2}(x) < y < \Gamma_\mu \}$.

Using Lemma 5 we obtain

**Corollary 1.** Let $(x;y)$ is an arbitrary point of the set $G$. Then the preimages of the point $(x;y)$ with respect to the restriction $F_\mu|_{K_1}$ does not belong to the set $D_{\mu,A_1} \cup T_{D_{\mu,A_1}}^{1} \cup T_{D_{\mu,A_1}}^{2}$.

By definitions of the sets $G$ and $T_{D_{\mu,2n-1}}^{1}, T_{D_{\mu,2n}}^{1}$ the following equality hold

$$
\bigcup_{n=1}^{+\infty} (T_{D_{\mu,2n-1}}^{1} \cup T_{D_{\mu,2n}}^{1}) = G.
$$

Our next step is the investigate of $F_\mu$ - trajectories of the points of the sets $T_{D_{\mu,2n-1}}^{1}$ and $T_{D_{\mu,2n}}^{1}$. For this goal, we consider the existence problem for implicit functions defined by the equations

$$
f_{\mu,n}(x, y) = \mu \tag{4}
$$

with initial condition

$$
f_{\mu,n}(x^*, y^*) = \mu \tag{5}
$$

on the interval $(\mu, +\infty)$, where $(x^*; y^*)$ is the point of one of the types described in Lemma 1. Using the existence theorem for a nonlocal $C^1$-smooth implicit function (see [7]) we obtain the following claim.

**Lemma 6.** Let $F_\mu be a map (1), \mu \in (0, 1]$. Then for every $n \geq 1$ there exists the $C^1$ – smooth function $y = \psi_{\mu,n}(x)$ which is the solution of the problem (4) with the initial condition (5) on the interval $(\mu, +\infty)$. Moreover, the graph $\psi_{\mu,n}$ of the function $y = \psi_{\mu,n}(x)$ passes through the triple point $M_{n-1}$ and the double point of the first type $M_{n+1}^{(2,1)}$ for $n \geq 1$; for $n \geq 2$ and $k \geq 3$ the graph $\psi_{\mu,n}$ of the function $y = \psi_{\mu,n}(x)$ passes through the double points of the second type $M_{n-k}^{(2,2)}$.

Let us note, that the ray $x = 2\mu$ for $\mu \in (0, 1]$ and its preimages of the different orders plays an important role in investigation of $F_\mu$ – trajectories from the set $G$.

Thus, we consider also the existence problem for implicit functions defined by the equations

$$
f_{\mu,n}(x, y) = 2\mu \tag{6}
$$
with initial conditions

\[ f_{\mu,n}(x_{Q_n}, y_{Q_n}) = 2\mu \]  

(7)

on the interval \((\mu, +\infty)\).

The point \(Q_n(x_{Q_n}; y_{Q_n})\) is the unique point of intersection of the connected component of the full preimage of the ray \(x = 2\mu\) of the \((n - 1)\) order with the unique smooth strictly monotone branch of the \(n\)th preimage of the ray \(x = \mu + 1\), which passes through the fixed point \(A_2(\mu + 1; 1)\) \((n \geq 1)\).

Let \(Q_1 = (2\mu, \frac{\mu + 1}{2\mu})\) is the unique point of intersection of the ray \(x = 2\mu\) with unique smooth strictly decreasing branch of the first preimage of the ray \(x = \mu + 1\), which passes through the fixed point \(A_2(\mu + 1; 1)\). Note that, for \(x > \mu\) there exist the unique point \(Q_2(\mu + \frac{\mu + 1}{2\mu}; \frac{2\mu}{\mu + \frac{\mu + 1}{2\mu}})\) of intersection of the first preimage of the ray \(x = 2\mu\) with the unique smooth strictly monotone branch of the second preimage of the ray \(x = \mu + 1\), which passes through the point \(A_2\) and the following equality hold \(F_\mu(Q_2) = Q_1\). Denote by \(Q_n\) \((n \geq 1)\) the unique point of intersection of connected component of the full preimage of the ray \(x = 2\mu\) of the \((n - 1)\) order with the unique continuos strictly monotone branch of the \(n\)th preimage of the ray \(x = \mu + 1\), which passes through the point \(A_2\) such that the following equality hold \(F_\mu(Q_{n+1}) = Q_n\) for any \(n \geq 1\). The point \(Q_n\) for any \(n \geq 1\) is the correctly defined.

Analogously above we apply the existence theorem of a nonlocal \(C^1\) – smooth implicit function [7] and obtain the following result.

**Lemma 7.** Let \(F_\mu\) be a map \((1), \mu \in (0, 1)\). Then there exists \(C^1\) – smooth monotone function \(y = \phi_{\mu,n}(x)\), which defined on the interval \((\mu, +\infty)\) by equality (6). The graph \(\phi_{\mu,n}\) of this function passes through the point \(Q_n\).

Lemmas 1, 6, 7 and definition of the set \(\{Q_n, n \geq 1\}\) makes it possible to construct the partition of the set \(T_{\Gamma_{\mu,n},2n-1} \subset G, n \geq 1\) such that boundaries of elements of this partition may contain the curves \(\eta_{\mu,n}, \phi_{\mu,m}, \psi_{\mu,k}\) (where \(m \geq 0, k \geq 0\) are defined by Lemma 1 and the set \(\{Q_n, n \geq 1\}\)), the critical line \(C_\mu\) and the ray \(x = 2\mu\).

For determination, we construct the partition of the set \(T_{\Gamma_{\mu,n},2n-1}\). Let \([a, b] \subset (\mu, \mu + 1)\) is an arbitrary closed interval. For \(n = 1\) we have

\[ T_{\Gamma_{\mu,n},1} = \{(x; y) \in \Gamma_{\mu,n} : \eta_{\mu,1}([a, b])(x) \leq y \leq \eta_{\mu,3}([a, b])(x)\}. \]

The critical line \(C_\mu\) and the line \(x = 2\mu\) intersect set \(T_{\Gamma_{\mu,n},1}\) in the double point of the first type \(M_1^{(2,1)}(x_{M_1^{(2,1)}}, y_{M_1^{(2,1)}})\) and in the point \(Q_1\) respectively, such that the equality hold

\[ T_{\Gamma_{\mu,n},1} = \bigcup_{i=1}^{3} T_{\Gamma_{\mu,n},1}^i, \]

where

\[ T_{\Gamma_{\mu,n},1}^1 = \{(x; y) \in T_{\Gamma_{\mu,n},1} : x \in (0, x_{M_1^{(2,1)})}; \]

\[ T_{\Gamma_{\mu,n},1}^2 = \{(x; y) \in T_{\Gamma_{\mu,n},1} : x \in (x_{M_1^{(2,1)}, x_{Q_1})}; \]

\[ T_{\Gamma_{\mu,n},1}^3 = \{(x; y) \in T_{\Gamma_{\mu,n},1} : x \in (x_{Q_1}, \mu + 1)\}. \]

From Lemmas 1, 6 follows, that the boundary of the set \(T_{\Gamma_{\mu,n},1}\) contains the triple point \(M_1^{(3)}(x_{M_1^{(3)}}, y_{M_1^{(3)}})\), which is the point of intersection of three branches: the monotone branch \(\eta_{\mu,1}\) of the first preimage of the line \(x = \mu + 1\) and the monotone branches \(\psi_{\mu,2}\) and \(\psi_{\mu,3}\) of
the second and the third preimages of the critical line $C_{\mu}$ respectively; in addition, the abscissa $x_{M_1^{(3)}}$ of the point $M_1^{(3)}$ satisfies inequality

$$\mu < x_{M_1^{(3)}} < 2\mu \text{ if } \mu \in (0, \frac{\sqrt{5} - 1}{2})$$

or satisfies inequality

$$2\mu < x_{M_1^{(3)}} < \mu + 1 \text{ if } \mu \in (\frac{\sqrt{5} - 1}{2}, 1).$$

For the case $\mu \in (0, \frac{\sqrt{5} - 1}{2})$ we define the sets

$$T_{D_{\mu}}^{2,1} = \{ (x; y) \in T_{D_{\mu}}^{n,1} : x \in (x_{M_1^{(2,1)}}, x_{M_1^{(3)})}, y < \psi_{\mu,3}(x) \};$$

$$T_{D_{\mu}}^{2,2} = \{ (x; y) \in T_{D_{\mu}}^{n,1} : x \in (x_{M_1^{(2,1)}}, x_{M_1^{(3)})}, \psi_{\mu,3}(x) < y < \psi_{\mu,2}(x) \};$$

$$T_{D_{\mu}}^{2,3} = \{ (x; y) \in T_{D_{\mu}}^{n,1} : x \in (x_{M_1^{(3)}}, x_{Q_1}), y > \psi_{\mu,2}(x) \},$$

such that $T_{D_{\mu}}^{2,i} = \bigcup_{i=1}^{3} T_{D_{\mu}}^{2,i}$. From this follows the equality

$$T_{D_{\mu}}^{n,1} = T_{D_{\mu}}^{1,1} \cup T_{D_{\mu}}^{3,1} \cup \bigcup_{i=1}^{3} T_{D_{\mu}}^{2,i} \text{ (see Figure 2)}$$

![Figure 2](image-url)

**Figure 2.** The partition of the set $T_{D_{\mu}}^{n,1}$ with using critical line $C_{\mu}$, ray $x = 2\mu$ and the monotone branches $\psi_{\mu,2}, \psi_{\mu,3}$ for $\mu \in (0, \frac{\sqrt{5} - 1}{2})$.

If $\mu \in (\frac{\sqrt{5} - 1}{2}, 1)$ we need the following sets

$$T_{D_{\mu}}^{2,1} = \{ (x; y) \in T_{D_{\mu}}^{n,1} : x \in (x_{M_1^{(2,1)}}, x_{Q_1}), y < \psi_{\mu,3}(x) \};$$
\[ T_{D_{2n,1}}^{2,3} = \{(x,y) \in T_{D_{2n,1}} : x \in (x_{M_{2}}, x_{Q_{1}}), \psi_{\mu,3}(x) < y < \psi_{\mu,2}(x)\}; \]

\[ T_{D_{2n,1}}^{2,1} = \{(x,y) \in T_{D_{2n,1}} : x \in (x_{M_{2}}, x_{Q_{1}}), y > \psi_{\mu,2}(x)\}; \]

\[ T_{D_{2n,1}}^{3,1} = \{(x,y) \in T_{D_{2n,1}} : x \in (x_{Q_{1}}, x_{M_{2}}), y < \psi_{\mu,3}(x)\}; \]

\[ T_{D_{2n,1}}^{3,2} = \{(x,y) \in T_{D_{2n,1}} : x \in (x_{Q_{1}}, x_{M_{2}}), \psi_{\mu,3}(x) < y < \psi_{\mu,2}(x)\}; \]

\[ T_{D_{2n,1}}^{3,3} = \{(x,y) \in T_{D_{2n,1}} : x \in (x_{Q_{1}}, \mu + 1), y > \psi_{\mu,2}(x)\}. \]

Then we obtain

\[ T_{D_{2n,1}}^{3,1} = T_{D_{2n,1}}^{3} \cup \left( \bigcup_{i=1}^{3} T_{D_{2n,1}}^{2,i} \right) \cup \left( \bigcup_{j=1}^{3} T_{D_{2n,1}}^{3,j} \right) \text{ (see Figure 3)}. \]

Further, we consider the second case in more detail.

Figure 3. The partition of the set \( T_{D_{2n,1}} \) with using critical line \( C_{\mu} \), ray \( x = 2\mu \) and the monotone branches \( \psi_{\mu,2}, \psi_{\mu,3} \) for \( \mu \in \left( \frac{\sqrt{3} - 1}{2}, 1 \right) \).

Let \( \mu \in \left( \frac{\sqrt{3} - 1}{2}, 1 \right) \). On the first step, describe above, we obtain the partition of the set \( T_{D_{2n,1}} \).

In this partition there exist elements, which contains the preimages of critical line \( C_{\mu} \) of the order \( k \geq 2 \) (for example, the set \( T_{D_{2n,1}}^{1} \) and the set \( T_{D_{2n,1}}^{3,3} \) are contain the connected components of the preimages of critical line \( C_{\mu} \) of the order 2 and 4 respectively, see Figure 3).

Suppose that after \( n \) steps, we constructed the partition of the set \( T_{D_{2n,1}}^{1} \) on the following subsets:

\[ T_{D_{2n-1}}^{2,1} = \{(x,y) \in T_{D_{2n-1}}^{2,1} : x \in (x_{M_{2}}, x_{Q_{2}}), y > \psi_{\mu,2n-1}(x)\}; \]

\[ T_{D_{2n-1}}^{3,2} = \{(x,y) \in T_{D_{2n-1}}^{3,2} : x \in (x_{Q_{2}}, x_{M_{2}}), \psi_{\mu,2n-1}(x) < y < \psi_{\mu,2n+1}(x)\}; \]
Thus, the partition of the set \( \mu \) branches

Let us note that every element of this partition can be contain the connected components of the preimages of critical line \( C_\mu \) of the order \( k \geq 2n \).

Let us describe \( n \)th step. Using definition of the points \( Q_n, n \geq 1 \), and definition of \( C_1 \) – smooth decreasing monotone functions \( y = \eta_\mu, (n), n \geq 1 \), we obtain that the monotone branches \( \psi, 2n \) and \( \phi, 2n \) intersects set \( T_{D_\mu}^{\mu, 2n+1} \); in addition, the following equality

\[
T_{D_\mu}^{\mu, 2n+1} = T_{D_\mu}^{1, 2n+1} \cup T_{D_\mu}^{2, 2n+1} \cup T_{D_\mu}^{3, 2n+1}
\]

hold, where the sets \( T_{D_\mu}^{1, 2n+1}, T_{D_\mu}^{2, 2n+1}, T_{D_\mu}^{3, 2n+1} \) are defined by equalities

\[
T_{D_\mu}^{1, 2n+1} = \{ (x; y) \in T_{D_\mu}^{\mu, 2n+1} : x \in (0, x_{M(2,1)}^{(n)}) \};
\]

\[
T_{D_\mu}^{2, 2n+1} = \{ (x; y) \in T_{D_\mu}^{\mu, 2n+1} : x \in (x_{M(2,1)}^{(n)}, x_{M(2,1)}^{(n)}) \};
\]

\[
T_{D_\mu}^{3, 2n+1} = \{ (x; y) \in T_{D_\mu}^{\mu, 2n+1} : x \in (x_{M(2,1)}^{(n)}, x_{M(2,1)}^{(n)}) \}.
\]

For all values of parameter \( \mu \) from the interval \( (\frac{\sqrt{5} - 1}{2}, 1) \) the inequalities are fulfilled \( x_{Q_2n+1} < x_{M(2,1)}^{(n)} < \mu + 1 \). Hence, the equality

\[
T_{D_\mu}^{\mu, 2n+1} = T_{D_\mu}^{1, 2n+1} \cup \bigcup_{i=1}^{3} T_{D_\mu}^{2, 2n+1} \cup \bigcup_{j=1}^{3} T_{D_\mu}^{3, 2n+1}
\]

holds, where

\[
T_{D_\mu}^{1, 2n+1} = \{ (x; y) \in T_{D_\mu}^{\mu, 2n+1} : x \in (x_{M(2,1)}^{(n)}), \psi, 2n(x) < y < \psi, 2n+3(x) \};
\]

\[
T_{D_\mu}^{2, 2n+1} = \{ (x; y) \in T_{D_\mu}^{\mu, 2n+1} : x \in (x_{M(2,1)}^{(n)}, x_{M(2,1)}^{(n)}), \psi, 2n(x) < y < \psi, 2n+3(x) \};
\]

\[
T_{D_\mu}^{3, 2n+1} = \{ (x; y) \in T_{D_\mu}^{\mu, 2n+1} : x \in (x_{M(2,1)}^{(n)}), \psi, 2n(x) < y < \psi, 2n+3(x) \};
\]

\[
T_{D_\mu}^{3, 3, 2n+1} = \{ (x; y) \in T_{D_\mu}^{\mu, 2n+1} : x \in (x_{M(2,1)}^{(n)}, x_{M(2,1)}^{(n)}), \psi, 2n(x) < y < \psi, 2n+3(x) \};
\]

Thus, the partition of the set \( T_{D_\mu}^{\mu, 2n-1} \) (for every \( n \geq 1 \)) is constructed.

Analogously above we construct the partition of the set \( T_{D_\mu}^{\mu, 2n} \) for every \( n \geq 1 \).

The properties of elements of the above partitions of the sets \( T_{D_\mu}^{\mu, 2n-1} \) and \( T_{D_\mu}^{\mu, 2n} \) for the case \( \mu \in (\frac{\sqrt{5} - 1}{2}, 1] \) are formulated in following claim.

**Lemma 8.** Let \( \mu \in (\frac{\sqrt{5} - 1}{2}, 1] \). Then map \( F_\mu \) is the diffeomorphism:
European Conference - Workshop "Nonlinear Maps and Applications"                                             IOP Publishing

(i) of the set $T^{2,i}_{D_{\Gamma_{\mu}}^n,2n}$ on the set $T^{2,i}_{D_{\Gamma_{\mu}}^n,2n-1}$, $i = 1, 2, 3$;
(ii) of the set $T^{3,j}_{D_{\Gamma_{\mu}}^n,2n}$ on the set $T^{3,j}_{D_{\Gamma_{\mu}}^n,2n-1}$, $j = 1, 2, 3$;
(iii) of the set $T^{1}_{D_{\Gamma_{\mu}}^n,2n}$ on the set $T^{1}_{D_{\Gamma_{\mu}}^n,2n-1}$.

Proof. For determination, we give the proof of the claim 1. As it follows from the definition of sets $T^{2,i}_{D_{\Gamma_{\mu}}^n,2n}$ and $T^{2,i}_{D_{\Gamma_{\mu}}^n,2n-1}$ ($i = 1, 2, 3$) the restriction $F_{\mu}|_{T^{2,i}_{D_{\Gamma_{\mu}}^n,2n}}$ is local diffeomorphism.

Let us prove, that $F_{\mu} : T^{2,i}_{D_{\Gamma_{\mu}}^n,2n} \rightarrow T^{2,i}_{D_{\Gamma_{\mu}}^n,2n-1}$ is surjection for every $i = 1, 2, 3$. In fact, by the formula (1) the connected component of the full preimage of the closed interval $\{(x,y) : y = c, x \in (x_1, x_2)\}$ contains the closed interval $\{(x,y) : x = c^*, y \in (y_1, y_2)\}$. Arbitrarily take and fix a leave $y = c$, where $c \in T^{2,i}_{D_{\Gamma_{\mu}}^n,2n-1}$, $c > 0$. From definition of the set $T^{2,i}_{D_{\Gamma_{\mu}}^n,2n-1}$ follows, that the abscises of the points $(x_1; c)$ and $(x_2; c) \in T^{2,i}_{D_{\Gamma_{\mu}}^n,2n-1}$ satisfy the inequalities $x_{M(2a+1)} < x_1 < x_2 < x_{M(2a+3)}$. Using formula (1), we obtain, that there exist the points $(c^*; y_1)$ and $(c^*; y_2) \in T^{2,i}_{D_{\Gamma_{\mu}}^n,2n}$, such that the equalities $F_{\mu}(c^*; y_1) = (x_1; c)$, $F_{\mu}(c^*; y_2) = (x_2; c)$ are fulfilled. It means, that $F_{\mu}$ is surjection of the set $T^{2,i}_{D_{\Gamma_{\mu}}^n,2n}$ on the set $T^{2,i}_{D_{\Gamma_{\mu}}^n,2n-1}$ for $i = 1, 2, 3$.

We show, that $F_{\mu} : T^{2,i}_{D_{\Gamma_{\mu}}^n,2n} \rightarrow T^{2,i}_{D_{\Gamma_{\mu}}^n,2n-1}$ is injection for every $i = 1, 2, 3$. Using the monotonicity of the function $f_{\mu}(x,y)$ by the variable $x$ (resp. $y$), we obtain, that for the arbitrary two points $(x_1; y_1)$, $(x_2; y_2) \in T^{2,i}_{D_{\Gamma_{\mu}}^n,2n}$, such that $x_1 < x_2$ (resp. $y_1 < y_2$) the inequality is fulfilled $F_{\mu}((x_1; y_1)) \neq F_{\mu}((x_2; y_2))$, where $F_{\mu}(x_1; y_1)$, $F_{\mu}(x_2; y_2) \in T^{2,i}_{D_{\Gamma_{\mu}}^n,2n-1}$ ($i = 1, 2, 3$). Hence, $F_{\mu} : T^{2,i}_{D_{\Gamma_{\mu}}^n,2n} \rightarrow T^{2,i}_{D_{\Gamma_{\mu}}^n,2n-1}$ is injection. Therefore, we obtain, that $F_{\mu} : T^{2,i}_{D_{\Gamma_{\mu}}^n,2n} \rightarrow T^{2,i}_{D_{\Gamma_{\mu}}^n,2n-1}$ is bijection ($i = 1, 2, 3$).

Then, the inverse map $F_{\mu}^{-1} : T^{2,i}_{D_{\Gamma_{\mu}}^n,2n-1} \rightarrow T^{2,i}_{D_{\Gamma_{\mu}}^n,2n}$ is defined, and the partial derivaties of its coordinate functions are continuous in $T^{2,i}_{D_{\Gamma_{\mu}}^n,2n-1}$. The claim 1 of Lemma 8 is proved.

Lemma 8 is proved.

By Lemmas 8 the following statement hold.

Corollary 2. The set $G$ is $F_{\mu}$ - invaraint set for every $\mu \in (0, 1]$.

II (c) In this part of the paper we describe asymptotic behaviour of $F_{\mu}$ - trajectories of the points of the set $G$.

Note that set $G$ contains periodic orbit $B$ of period 2. Prove the following statement.

Lemma 9. The map $F_{\mu}$ have the unique periodic orbit $B$ with the least period two, which formed by two sources $B_1 \left(\frac{\mu^2+1-\sqrt{\mu^2+1}}{\mu}, \frac{\mu^2}{(1-\sqrt{\mu^2+1})^2}\right)$ and $B_2 \left(\frac{\mu^2+1+\sqrt{\mu^2+1}}{\mu}, \frac{\mu^2}{(1+\sqrt{\mu^2+1})^2}\right)$ such that $B_1 \in T^{1}_{D_{\Gamma_{\mu}}^n,1}$, $B_2 \in T^{1}_{D_{\Gamma_{\mu}}^n,2}$.

Proof.

(i) Existence of unique periodic orbit $B$ is proved in [8].
(ii) Let us prove, that $B_1 \in T^{1}_{D_{\Gamma_{\mu}}^n,1}$. Note, that for any $\mu \in (0, 1]$ for the coordinate of point $B$ the equalities holds:

$$\frac{\mu^2+1-\sqrt{\mu^2+1}}{\mu} = \frac{(\sqrt{\mu^2+1}-1)\sqrt{\mu^2+1}}{\mu} = \frac{\mu\sqrt{\mu^2+1}}{\sqrt{\mu^2+1}+1}$$
and
\[
\frac{\mu^2}{(1 - \sqrt{\mu^2 + 1})^2} = \frac{(1 + \sqrt{1 + \mu^2})^2}{\mu^2} = \left(\frac{1}{\mu} + \sqrt{1 + \frac{1}{\mu^2}}\right)^2.
\]

Then the following inequalities are fulfilled
\[
0 < \frac{\mu \sqrt{\mu^2 + 1}}{\mu^2 + 1 + 1} < \mu, \quad \left(\frac{1}{\mu} + \sqrt{1 + \frac{1}{\mu^2}}\right)^2 > 1.
\]

Moreover, for any \(\mu \in (0, 1]\) we have:
\[
\frac{\mu^2 + 1 - \sqrt{\mu^2 + 1}}{\mu} - \mu - 1 = \mu \sqrt{\mu^2 + 1} - \mu - 1 = -1 + \sqrt{1 + \frac{1}{\mu^2}} + \frac{1}{\mu} > 0.
\]

Hence, we obtain that \(f_{\mu,1}(x,y)|_{B_1} > \mu + 1\).

Using the last formula and the inequalities (8), we obtain, that point \(B_1\) belongs to the unbounded curvilinear sector, which boundary formed by ray \(\{(x,y) : x = \mu, y > \frac{\mu + 1}{\mu}\}\) and by unbounded continuous arc \(\{(x,y) : 0 < x < \mu, y = \frac{\mu + 1}{x}\}\). Consequently, \(B_1 \in T_{D_{\mu^2}}^1\) (see definition of set \(T_{D_{\mu^2}}^1\)). Using Lemma 6 for any \(\mu \in (0, 1]\) we obtain, that \(B_2 \in T_{D_{\mu^2}}^2\).

Lemma 9 is proved.

By Lemma 9 and Corollary 2 the preimages of periodic orbit \(B\) of the arbitrary order \(n \geq 1\) (with respect to restriction \(F_{\mu,K_1}\)) belong to the set \(\bigcup_{n=1}^{+\infty} (T_{D_{\mu^2}}^1 \cup T_{D_{\mu^2}}^2 \cup T_{D_{\mu^2}}^3 \cup \cdots)\).

Denote by \(\{(F_{\mu,K_1})^{-n}(x_{B_1},y_{B_1})\}\) the subset of the full \(n\)th preimage of the point \(B_1\) which does not contain the periodic orbit \(B\). From Lemmas 8, 9 follows, that \(\{(F_{\mu,K_1})^{-n}(x_{B_1},y_{B_1})\}\) is single point set. The unique point of this set we denote by \(x_{B_1},y_{B_1}\).

By the claim 3 of Proposition 3, Lemmas 8, 9 and Corollary 2 the following statement holds.

**Lemma 10.** Let \(F_{\mu}\) be a map of type \((1), \mu \in (0, 1]\). Then for any \(\mu \in (0, 1]\) the limit equalities
\[
\lim_{n \to +\infty} x_{B_1} = \mu + 1, \quad \lim_{n \to +\infty} y_{B_1} = 1.
\]

are valid.

Note, that the trajectory \(\{(F_{\mu,K_1})^{-2n}(x_{B_1},y_{B_1})\}_{n \in \mathbb{Z}}\) (\(\{(F_{\mu,K_1})^{-2n}(x_{B_1},y_{B_1})\}_{n \in \mathbb{Z}}\)) is heteroclinic, such that it is attracted to the point \(B_1\) and it is repelled from the point \(A_2\).

By definition of sets \(T_{D_{\mu^2}}^1\) and \(T_{D_{\mu^2}}^2\) the following inclusion
\[
F_{\mu}(T_{D_{\mu^2}}^1) \subset T_{D_{\mu^2}}^2
\]
holds.

From Lemmas 5 and 8 we obtain the following statement.

**Corollary 3.** The set \(D_{\mu,A_2}\) is invariant set.

Using inclusion (9) and Lemmas 4, 5, 8–10 we obtain the following result.

**Lemma 11.** Let \(F_{\mu}\) be a map of type \((1), \mu \in (0; 1]\). Then for every point \((x,y) \in G\) with exception of periodic orbit \(B\) of period two and all of its preimages the equalities hold
\[
\lim_{n \to +\infty} f_{\mu,n}(x,y) = 0, \quad \lim_{n \to +\infty} g_{\mu,n}(x,y) = \mu^2.
\]

Claim 2 of Proposition 4 follows immediately from Lemmas 5, 11 and Corollary 3.

Proposition 4 is proved.
4. Completion of the proof of Theorem 1
In this part we describe nonwandering set $Ω(F_µ)$ of map $F_µ$ and complete the proof of Theorem 1.

From Proposition 4 follows, that for any $µ ∈ (0, 1)$ nonwandering set $Ω(F_µ|K_1)$ of the map $F_µ|K_1$ is finite and consists of the fixed point $A_2(µ + 1; 1)$ and the periodic orbit $B$ of period two.

Let us describe behaviour of $F_µ$ – trajectories in $K_2$. As it follows from the claims 1 and 2 of Proposition 1, $F_µ|K_2 : K_2 → F_µ(K_2)$ is diffeomorphism.

By the claim 4 of Proposition 4 we obtain, that for any $µ ∈ (0, 1)$ the set $K_2$ contains two fixed points: the saddle point $A_3(µ − 1; 1)$ and the sink $A_1(0; µ^2)$. Therefore, there exist continuous functions $γ_i^n, γ_i^s, i = 1, 2$, such that the graphs $γ^n_1, γ^n_2$ are the nonstable separatrix of saddle point $A_3$ and the graphs $γ^s_1, γ^s_2$ are the stable separatrix of saddle point $A_3$ [14]. Moreover, the equalities holds $γ^n_1 ∪ γ^n_2 = W^s(A_3), γ^s_1 ∪ γ^s_2 = W^u(A_3)$. Let us define the sets

$$K_{21} = \{(x, y) ∈ K_2 : y > γ^n_1(x), y > γ^n_2(x)\};$$

$$K_{22} = \{(x, y) ∈ K_2 : γ^n_2(x) < y < γ^n_1(x)\};$$

$$K_{23} = \{(x, y) ∈ K_2 : y < γ^s_2(x), y < γ^s_1(x)\};$$

$$K_{24} = \{(x, y) ∈ K_2 : γ^s_2(x) < y < γ^s_1(x)\},$$

such that $\bigcup_{i=1}^4 K_{2i} = K_2$.

Lemma 12. Let $F_µ$ be a map of type (1), $µ ∈ (0, 1)$. Then

(i) for any point $(x, y) ∈ K_{22} ∪ K_{23}$ the following equalities holds

$$\lim_{n→+∞} f_{µ,n}(x, y) = 0, \quad \lim_{n→+∞} g_{µ,n}(x, y) = µ^2;$$

(ii) for any point $(x, y) ∈ K_{21} ∪ K_{24}$ the following equalities holds

$$\lim_{n→+∞} f_{µ,n}(x, y) = −∞, \quad \lim_{n→+∞} g_{µ,n}(x, y) = +∞.$$

Let us describe the nonwandering set $Ω(F_µ|K_2)$ of the map $F_µ|K_2$, $µ ∈ (0, 1)$.

From Lemma 12 follows, that for any $µ ∈ (0, 1)$ set $Ω(F_µ|K_2)$ is finite and consists of fixed points $A_3(µ − 1; 1), A_1(0; µ^2)$; for $µ = 1$ the equality is fulfilled $A_3(µ − 1; 1) = A_1(0; µ^2)$ and the set $Ω(F_1|K_2)$ consists of the single point $A_1(0; 1)$.

By formula (1) we obtain, that for any $µ ∈ (0, 1]$ the following equalities holds

$$F_µ(Ox) = \{0\} × [0, +∞), \quad F_µ(Oy) = (0; µ^2).$$

Therefore, for any $µ ∈ (0, 1]$ the equality is fulfilled

$$(Ox ∪ Oy) ∩ Ω(F_µ) = A_1(0; µ^2).$$

In addition, for any $µ ∈ (0, 1)$ the global stable manifold $W^s(A_1)$ of sink $A_1$ and the global unstable manifold $W^u(A_3)$ of saddle point $A_3$ intersect transversally (see Figure 4). Therefore, we obtain, that for any $µ ∈ (0, 1)$ the map $F_µ$ is a (nonsingular) Morse-Smale endomorphism. For $µ = 1$ there exist the nonhyperbolic point $A_1(0; 1)$ and map $F_1$ is the singular Morse-Smale endomorphism.

Thus, Theorem 1 is proved.

Acknowledgments
The author thanks Professor Efremova L. S. for helpful discussions.
Figure 4. Asymptotic behaviour of $F_\mu$ – trajectories in $K_2$.

References

[1] C Mira 1987 Chaotic dynamics from the One-Dimensional Endomorphism to the Two-Dimensional Diffeomorphism (Singapore: World Scientific)


[8] S S Belmesova, L S Efremova 2013 A one-parameter family of quadratic maps of a plane including Morse Smale endomorphisms Russian Math. 8 70–74

[9] S Belmesova, L Efremova 2010 On Quadratic Maps from Some One-Parameter Family Closed to Unperturbed Map Proceed. of MIPT 2 (2) 46–57 (Russian)

[10] S S Belmesova, L S Efremova, D Fournier-Prunaret 2014 Invariant curves of quadratic maps on the plane from the one-parameter family containing the trace map ESAIM: Proceedings and Surveys 46 98-110


