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Dynamics of one model of the fast kinematic dynamo

Timur Medvedev¹, Vladislav Medvedev² and Evgeny Zhuzhoma²

¹ National Research University Higher School of Economics, 136 Rodionova Street, Nizhny Novgorod, Russia
² National Research University Higher School of Economics, 25/12 Bolshaya Pecherckaya Street, Nizhny Novgorod, Russia
E-mail: tmedvedev@hse.ru

Abstract. We suggest a new model of the fast nondissipative kinematic dynamo which describes the phenomenon of exponential growth of the magnetic field caused by the motion of the conducting medium. This phenomenon is known to occur in the evolution of magnetic fields of astrophysical bodies. In the 1970s A.D. Sakharov and Ya.B. Zeldovich proposed a “rope” scheme of this process which in terms of the modern theory of dynamical systems can be described as Smale solenoid. The main disadvantage of this scheme is that it is non-conservative. Our model is a modification of the Sakharov-Zeldovich’s model. We apply methods of the theory of dynamical systems to prove that it is free of this fault in the neighborhood of the nonwandering set.

1. Introduction

One of the fundamental problems of the natural sciences is formation and evolution of magnetic fields of astrophysical bodies. Particularly, the theory of the kinematic dynamo studies the evolution of the magnetic fields of electrically conducting fluids [1, 2, 3]. The velocity field \( \vec{v} \) of an incompressible conducting medium (fluid, gas or plasma) is supposed to be given while the subject of interest being the magnetic field \( \vec{H} \) stretched by the fluid flow in the presence of a low diffusion dissipating the magnetic energy. The kinematic dynamo is described by the following equations

\[
\frac{\partial \vec{H}}{\partial t} = \text{rot} \left[ \vec{v} \vec{H} \right] + \eta \nabla \times \vec{H}, \quad \text{div} \, \vec{H} = 0, \quad \text{div} \, \vec{v} = 0,
\]

here \( \eta \) is the magnetic diffusivity which is in inverse proportion to Magnetic Reynolds number \( R_m = \frac{1}{\eta} \) (see the main notions and definitions in [4, 5, 6]). The literature on the magnetohydrodynamics often uses \( \vec{B} \) to describe the magnetic field, where \( \vec{B} = \mu \vec{H} \) and \( \mu \) is the permeability of the medium (for us the difference between \( \vec{H} \) and \( \vec{B} \) is irrelevant). One of the important aspects of the kinematic dynamo is the fast kinematic dynamo when the motion of the conducting medium causes an exponential growth of the magnetic field for a small magnetic diffusion.

The discrete (in time) version of this problem studies the growth of the magnetic field at moments \( t = 1, 2, \ldots \). Instead of the transport of the flow and the continuous diffusion of the magnetic fields one considers the composition of these processes. That is, for a given conservative (volume preserving) diffeomorphism \( f : M \rightarrow M \) the magnetic field is considered to
be first transported to the field \( f_\nu(\vec{H}) \) and then to be dissipated as the solution of the equation
\[
\frac{\partial f_\nu(\vec{H})}{\partial t} = \eta \Delta \left( f_\nu(\vec{H}) \right).
\]

A kinematic dynamo is said to be dissipative (“realistic”) if \( \eta \to +0 \) or nondissipative (“idealistic”) for \( \eta = 0 \). For the nondissipative dynamo the magnetic field is “frozen” into the movement of the medium \([7]\) and one usually studies the exponential growth of the energy of this field. According to \([8]\) a fast nondissipative dynamo occurs if the diffeomorphism movement of the medium \([7]\) and one usually studies the exponential growth of the energy of this field.

Consider the direct product \( K \times [0; 1] \), where \( K = [-1; +1] \times [-1; +1] \) is the square on the plane \( \mathbb{R}^2 \) with the Cartesian coordinates \((x, y)\). Let \( R_t : \mathbb{R}^2 \to \mathbb{R}^2 \) denote the counterclockwise rotation
\[
\begin{align*}
\bar{x} &= x \cos \pi t - y \sin \pi t \\
\bar{y} &= x \sin \pi t + y \cos \pi t
\end{align*}
\]
of the plane \( \mathbb{R}^2 \) through the angle \( \pi t \). The set \( \bigcup_{0 \leq t \leq 1} (t, R_t(K)) \) is homeomorphic to \( K \times [0; 1] \) because \( R_t \) is a homeomorphism for each \( t \). Since \( R_t(K) = K \), the squares \( K \times \{0\}, K \times \{1\} \) can be naturally identified by \( id : K \times \{0\} \to K \times \{0\} \). Let \( B \) be the body \( \bigcup_{0 \leq t \leq 1} (t, R_t(K)) \) with the squares \( K \times \{0\} \) and \( K \times \{1\} \) identified by \( id \):
\[
\bigcup_{0 \leq t \leq 1} (t, R_t(K)) / \left( K \times \{1\} \sim K \times \{0\} \right) \overset{\text{def}}{=} B.
\]

We assume that the identification \( id \) reverses the orientation if the initial orientation of the squares \( K \times \{0\}, K \times \{1\} \) is induced by an arbitrary orientation of the body \( \bigcup_{0 \leq t \leq 1} (t, R_t(K)) \). The body \( B \) is a twisted cylinder shown in Fig. 1, where (a) shows the part of \( B \) for the values \( 0 \leq t \leq \frac{1}{2} \) while (b) shows it for \( \frac{1}{2} \leq t \leq 1 \).
It is clear that the set
\[ \bigcup_{0 \leq t < 1} (t, (0, 0)) \overset{\text{def}}{=} S_0^1 \]
is a circle on which the quotient map \([0; 1] \to [0; 1]/(0 \sim 1) = S^1\) induces the cyclic coordinate \(t \mod 1\). We say the circle \(S_0^1\) to be the axis of the body \(B\). We embed \(B\) into \(\mathbb{R}^3\) in such a way that the axis \(B\) has no knots in \(\mathbb{R}^3\) and we consider \(B\) to be identical to its embedding. First we are going to construct the desired diffeomorphism \(F : B \to f(B) \subset \mathbb{R}^3\) of the body \(B\) onto its image in some neighborhood homeomorphic to the solid torus. Notice that the natural projection \(K \times [0; 1] \to [0; 1]\) is a trivial bundle and it induces the locally trivial bundle \(p_1 : B \to S_0^1\) with the fiber \(K\). Let \(D_t\) denote the fiber over \(t \in S_0^1\), \(D_t = p_1^{-1}(t)\). Evidently \(D_t\) can be considered as \(R_t(K)\), i.e. as the result of the rotation \(R_t\) of the square \(K\).

To define the diffeomorphism \(F\) we need to modify the map introduced by Stephen Smale known as the Smale horseshoe \([11, 9]\). Recall that the classic Smale horseshoe is a diffeomorphism of some disk, containing the square \(K = D_0^2\) on the plane \(\mathbb{R}^2\), into itself. The diffeomorphism \(w : D_0^2 = D^2 \to \mathbb{R}^2\) of this square is the composition of a contraction along the axis \(Ox\), an expansion along the axis \(Oy\), a bend (the direction of the bend is irrelevant) of the resulting rectangle and, finally, its translation in such a way that the intersection \(D^2 \cap w(D^2)\) is the union of two disjoint strips which are symmetric with respect to the axis \(Oy\)\(^1\) (see Fig. 2(a)). Clearly, the contraction and the expansion could be chosen in such a way that the Jacobian determinant \(J(w)\) of \(w\) on \(D^2\) equals \(\frac{1}{2}\). From now on we suppose these conditions to be satisfied. Denote by \(sh_0 : \mathbb{R}^2 \to \mathbb{R}^2\) the translation \((x; y) \mapsto (x + \frac{1}{2}; y)\) along the axis \(Ox\) and let \(w_0 = sh_0 \circ w : D^2 \to \mathbb{R}^2\). Let \(S_0 : \mathbb{R}^2 \to \mathbb{R}^2\) denote the inversion with respect to the origin \((0; 0)\), \(S_0(x; y) = (-x; -y)\). Again one can pick the contraction, the expansion and the bend such that the following conditions are satisfied (see Fig. 2(b) on the right):

(i) the intersection \(D^2 \cap w_0(D^2)\) consists of two disjoint strips;
(ii) the sets \(w_0(D^2), S_0(w_0(D^2))\) are disjoint.

\(^1\) The horseshoe is sometimes defined as the diffeomorphism of the square which is afterwards extended to the entire plane. It is known \([12, 11]\) that \(w\) can be extended to a map of the entire plane \(\mathbb{R}^2\) in such a way that this map is the identity outside some neighborhood of \(D^2\).
This contradicts to (1).

The first condition means that the map \( s_0 \circ w \) is a Smale horseshoe whose symmetry line is perpendicular to the axis \( Ox \). The second condition means that the horseshoe \( \omega_0(D^2) \) is disjoint from its reflection by \( S_0 \). Notice that \( S_0 \circ \omega_0(D^2) \) is a horseshoe as well.

Consider a neighborhood \( S^1 \times B^2 \) of the body \( B \) homeomorphic to the solid torus; here \( B^2 \subset \mathbb{R}^2 \) is a disk large enough containing the square \( K \) and \( S^1 \) is a circle with cyclic coordinate \( t \mod 1 \). Below we identify a neighborhood \( B \) of the square \( K \) with the natural parametrization \( [0; 1) \). We now define the map \( F \) as the result of rotation \( R_t(K) \) of the square \( K \), therefore we can define a Smale horseshoe on \( D_t \). Let

\[
\omega_{0t} = R_t \circ \omega_0 \circ R_{-t} : D_t \to \{t\} \times B^2.
\]

This map forms the horseshoe in the direction of the line \( y = x \cdot \tan \pi t \) only when the symmetry line of the horseshoe \( \omega_{0t}(D^2) \) is perpendicular to the line \( y = x \cdot \tan \pi t \).

Let \( S^1 = [0; 1]/(0 \sim 1) \) be a circle with the natural parametrization \( [0; 1] \to [0; 1]/(0 \sim 1) = S^1 \). The map \( E_2 : S^1 \to S^1 \) of the form \( t \to 2t \mod 1 \) is an expanding endomorphism of the circle of degree 2 [15]. We now define the map \( F : B \to S^1 \times B^2 \) in the following way: for every \( t \in [0; 1) \) and every \( z \in D_t \) let

\[
(t; z) \mapsto (E_2(t); R_t \circ \omega_{0t}(z)), \quad t \in [0; 1), \quad z \in D_t.
\]

Notice that from the definition of \( F \) if follows that \( F(D_t) \subset B_{2t \mod 1} \), (Fig. 3).

**Lemma 1** The map \( F : B \to F(B) \subset S^1 \times B^2 \) is a diffeomorphism onto its image.

**Proof** Assume \( F(t_1; z_1) \cap F(t_2; z_2) \neq \emptyset \), then \( F(D_{t_1}) \cap F(D_{t_2}) \neq \emptyset \). From the definition of \( F \) it follows that \( E_2(t_2) = E_2(t_1) \), i.e. \( 2t_1 \mod 1 = 2t_2 \). Since the map \( \omega_{0t} \) is a diffeomorphism to its image one assumes \( t_1 \neq t_2 \), therefore \( t_2 = t_1 + \frac{1}{2} \). Then \( F(D_{t_1}) = R_{2t_1} \circ \omega_{0t} \circ R_{-t_1}(D_{t_1}) \)

\[
F(D_{t_2}) = F(D_{t_1} + \frac{1}{2}) = R_{2t_1+1} \circ \omega_0 \circ R_{-t_1-\frac{1}{2}}(D_{t_1}) = R_1 \circ R_{2t_1} \circ \omega_0 \circ R_{-t_1-\frac{1}{2}}(D_{t_1}).
\]

Since \( R_1 \) is the rotation through \( \pi \), the horseshoes \( F(D_{t_1}) \) and \( S_0 \circ F(D_{t_1}) \) must intersect and this contradicts to (1).\( \square \)
Notice that since the Jacobian determinant $J(w)$ of the map $w$ on $D^2$ equals $\frac{1}{2}$, the Jacobian determinant of $F$ equals $J(F) = J(w) \cdot DE_2 = \frac{1}{2} \cdot 2 = 1$ and therefore, $F$ is a conservative diffeomorphism to its image. The union $\mathbb{R}^3 \cup \{\infty\}$ of the Euclidean space and the point at infinity $\{\infty\}$ can be identified with the 3-sphere $S^3$ in the standard way.

**Lemma 2** The map $F : B \to F(B) \subset S^1 \times B^2 \subset \mathbb{R}^3$ can be extended to a diffeomorphism $f : S^3 \to S^3$ which is conservative in some neighborhood of $B$.

**Proof** By construction the circle $S^1_0$ is the axis of the solid torus $S^1 \times B^2$ and the body $B$ as well as $S^1 \times B^2$ are its tubular neighborhoods. The diffeomorphism of the square in the Smale horseshoe can be extended to a diffeomorphism of a disk large enough (see [12]), therefore $F$ can be extended to a diffeomorphism (which we again denote by $F$) of the solid torus $S^1 \times B^2$ which preserves the disk structure. Without loss of generality one assumes that $F$ is conservative in some neighborhood of $B$ (otherwise one can consider the square $K$ a bit larger). If follows from the construction that the curves $S^1_0$ and $F(S^1_0)$ are knot free in $\mathbb{R}^3$. Therefore there is a deformation of $S^1_0$ to $F(S^1_0)$ which can be easily extended to a deformation of their tubular neighborhoods $\phi_0 : S^1 \times B^2 \to F(S^1 \times B^2)$. Clearly $\phi_0$ can be made conservative. It follows from [16] that $\phi$ can be extended to the desired diffeomorphism $f : S^3 \to S^3$. $\square$

3. The dynamics of the nonwandering set

We say the solid torus $B$ embedded into $S^3$ to be basic and we denote it by $B$. Let

$$\Omega = \bigcap_{n=-\infty}^{\infty} f^n(B).$$

The set $\Omega$ is invariant with respect to $f$ ([12]) and it is not empty because it contains in $D_0 = \{0\} \times D^2 \subset B$ the invariant nontrivial (0-dimensional) set $\Omega_0$ of the Smale horseshoe ([12, 11, 9]). Let $Diff^1(S^3)$ denote the space of diffeomorphisms of the 3-sphere $S^3$ with $C^1$ topology.

**Lemma 3** The set $\Omega$ is hyperbolic and the restriction $f|_\Omega$ of $f$ to $\Omega$ is of positive topological entropy. Moreover, there is a neighborhood $U(f)$ of $f$ in the space $Diff^1(S^3)$ such that every diffeomorphism $g \in U(f)$ has a hyperbolic invariant set $\Omega_g \subset B$, the diffeomorphisms $f|_\Omega, g|_{\Omega_g}$ are conjugate and the entropy of the restriction $g|_{\Omega_g}$ is positive.

**Proof** By construction the Jacobian determinant of the map $f|_B : S^1 \times D^2 \to \mathbb{R}^3$ is equal to $J(f) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$. Therefore $f$ is hyperbolic (not only on $\Omega$ but on $B$ as well) and it follows that
the set $\Omega$ is hyperbolic. The restriction $f|_{t_0}: \Omega_0 \to \Omega_0$ has a positive entropy ([12, 13]). Then it follows from [17] that the entropy of the restriction $f|_\Omega$ is positive as well. Since hyperbolic sets are stable under $C^1$-small perturbations, the desired neighborhood $U(f)$ exists because the entropy is invariant with respect to conjugacy. □

Now we study the dynamics of the restriction of the diffeomorphism $f : S^4 \to S^4$ to its nonwandering set belonging to the basic solid torus $B$. To do this we construct a symbolic model of $f|_\Omega$ on the invariant set $\Omega$. Let $t_0 \in S^1, \, 0 \leq t_0 < 1$ be fixed. Consider the intersection of $\Omega$ and the disk $D_{t_0} = \{t_0\} \times B^2 \subset S^1 \times B^2$.

Following the symbolic model of the classic Smale horseshoe we define two vertical and two horizontal (in the usual sense) strips in the square $D_{t_0}$ in the following way. Recall that the intersection of the square $D_{t_0}$ with its image with respect to the map of the Smale horseshoe $w_{t_0}$ consists of the two vertical strips,

$$w_{t_0}(D_{t_0}) \cap D_{t_0} = R_0(t_0) \cup R_1(t_0),$$

where $R_0(t_0)$ (respectively $R_1(t_0)$) is the strip nearest to the center (respectively, the farthest). It follows from the construction that in $D_{t_0}$ there are two disjoint horizontal (perpendicular to $R_0(t_0), \, R_1(t_0)$) strips (we denote them by $R_0^{(-1)}(t_0)$ and $R_1^{(-1)}(t_0)$) such that

$$w_{t_0}(R_0^{(-1)}(t_0)) = R_0(t_0) \quad \text{and} \quad w_{t_0}(R_1^{(-1)}(t_0)) = R_1(t_0). \quad (3)$$

We now show that the intersection $f^{-1}(B) \cap D_{t_0} \cap f(B)$ consists of eight rectangles. By construction the intersection $D_{t_0} \cap f(B) \cap B = D_{t_0} \cap f(B)$ consists of four strips. Indeed, there are exactly two points $t_1' = \frac{t_0}{2}, \, t_2' = \frac{t_0}{2} + \frac{1}{2} \in S^1$ such that $t_0 = E_2(t_1')$ and $t_0 = E_2(t_2')$ and $D_{t_0} \cap f(D_{t_0}) = \emptyset$ for every $\mu \neq t_1', t_2'$, $0 \leq \mu < 1$. Then

$$D_{t_0} \cap f(B) = D_{t_0} \cap \left( f(D_{t_1'}) \cup f(D_{t_2'}) \right) = \left( D_{t_0} \cap f(D_{t_1'}) \right) \cup \left( D_{t_0} \cap f(D_{t_2'}) \right).$$

Each intersection $D_{t_0} \cap f(D_{t_1'})$, $D_{t_0} \cap f(D_{t_2'})$ consists of two strips. Notice that $D_{t_0} \cap f(D_{t_1'}) = R_0(t_0) \cup R_1(t_0)$. Then by the definition of $f$

$$f(R_i(t_0)) = R_i(E_2(t_0)), \quad i = 1, 2. \quad (4)$$

Applying (3) with $t_0$ changed to $E_2(t_0)$ and (4) we get

$$R_0^{(-1)}(t_0) \cap R_1^{(-1)}(t_0) = f^{-1}(R_0(E_2(t_0)) \cup R_1(E_2(t_0))).$$

Therefore the intersection $f^{-1}(B) \cap D_{t_0}$ consists of two horizontal strips $R_0^{(-1)}(t_0) \cap R_1^{(-1)}(t_0)$. Then the intersection $f^{-1}(B) \cap D_{t_0} \cap f(B)$ consists of 8 rectangles which we say to be rectangles of the first degree. They are the intersection of four vertical strips and two horizontal strips, see Fig. 4 (a).

Similar to the standard process of coding for the classic Smale horseshoe we code the rectangles of first degree in the following way. Recall that $D_{t_0} \cap f(D_{t_0})$ is a two vertical strips and each strip divides the disk. We assign “0” to the strip of $D_{t_0} \cap f(D_{t_0})$ which is the closest to the coordinate origin and we assign “+1” to the other strip. In the same way we assign “0” and “+1” to the two strips of the intersection $D_{t_0} \cap f(D_{t_0})$. Notice that $D_{t_0} \cap f(D_{t_0}) = R_0(t_0) \cup R_1(t_0)$ and we have assigned “0” to the strip $R_0(t_0)$ and we have assigned “+1” to the strip $R_1(t_0)$. We assign $\omega_0 = 0$ and $\omega_0 = 1$ to the respective horizontal strips $R_0^{(-1)}(t_0)$ and $R_1^{(-1)}(t_0)$. Now each rectangle of the first degree has the corresponding block $[t_0, \omega_0; t_1, \omega_1]$ where $t_0 = E_2(t_1)$, $\omega_0 \in \{0; 1\}$, $\omega_1 \in \{0; 1\}$. Denote by $V(1)\{[t_0, \omega_0; t_1, \omega_1]\}$ the rectangle with this block. Since
Figure 4. Rectangles of the first degree (a), rectangles of the second degree (b).

the rectangles of the first degree are pairwise disjoint it follows from (2) that an arbitrary point $P \in \Omega \cap D_{t_0}$ belongs to exactly one octagon and $P \in V^{(1)}[(t_0, \omega_0); (t_1, \omega_1)]$ has been assigned the (initial) block $[(t_0, \omega_0); (t_1, \omega_1)]$ of the first degree.

Analogously the intersection

$$f^{-2}(B) \cap f^{-1}(B) \cap D_{t_0} \cap f(B) \cap f^2(B) = f^{-1} [f^{-1}(B) \cap B] \cap D_{t_0} \cap f [B \cap f(B)]$$

consists of $8^2$ rectangles which we call the rectangles of the second degree, see Fig. 4 (b). It is easy to see that each rectangle of the first degree contains 8 disjoint rectangles of the second degree. Using the rectangle of the first degree $V^{(1)}[(t_0, \omega_0); (t_1, \omega_1)]$ instead of the disk we analogously assign the block $V^{(2)}[(t_0, \omega_0); (t_1, \omega_1); (t_2, \omega_2)]$ to the rectangle of the second degree $V^{(1)}[(t_0, \omega_0); (t_1, \omega_1)]$, where $t_1 = E_2(t_0), t_2 = E_2(t_2), \omega_j \in \{0; 1\}, j = -1, 0, 1, 2$. Since a point $P \in \Omega \cap D_{t_0}$ is contained in exactly one octagon of the second degree, say $V^{(2)}[(t_1, \omega_1); (t_2, \omega_2)]$, the unique block of the second degree $[(t_1, \omega_1); (t_2, \omega_2)]$ is assigned to it. If we continue this procedure then for each point $P \in \Omega \cap D_{t_0}$ we get the bilateral sequence

$$\hat{P} = [\cdots (t_{-n}, \omega_{-n}); \cdots ; (t_{-1}, \omega_{-1}); (t_0, \omega_0); (t_1, \omega_1); (t_2, \omega_2); \cdots ; (t_n, \omega_n); \cdots],$$

where $\omega_j \in \{0; 1\}, j \in \mathbb{Z}, E_2(t_{i+1}) = t_i, i \in \mathbb{Z}$. The underlined pair is conventionally assumed to be at the position 0.

Let $\Sigma_2(E_2)$ denote the set of all sequences of the type

$$[\cdots (t_{-n}, \omega_{-n}); \cdots ; (t_{-1}, \omega_{-1}); (t_0, \omega_0); (t_1, \omega_1); (t_2, \omega_2); \cdots ; (t_n, \omega_n); \cdots],$$

where $\omega_j \in \{0; 1\}, j \in \mathbb{Z}, E_2(t_{i+1}) = t_i, i \in \mathbb{Z}$. Fix a sequence $\hat{P}^{(0)} \in \Sigma_2(E_2)$, $\hat{P}^{(0)} = \{ (t_i^{(0)}, \omega_i^{(0)}) \}_{i=-\infty}^{\infty}$ and fix numbers $r \in \mathbb{N}, \varepsilon > 0$. A $(r, \varepsilon)$-neighborhood $U_{r, \varepsilon}(\hat{P}^{(0)})$ of the sequence $\hat{P}^{(0)}$ is the set of sequences $\hat{P} \in \Sigma_2(E_2)$, $\hat{P} = \{ (t_i, \omega_i) \}_{i=-\infty}^{\infty}$ which satisfy

$$|t_i^{(0)} - t_i| < \varepsilon \text{ for all } -r \leq i \leq r, \sum_{i=-\infty}^{\infty} \frac{|\omega_i^{(0)} - \omega_i|}{2^i} < \varepsilon.$$
The set of \((r, \varepsilon)-\)neighborhoods generates the topology on \(\Sigma_2(E_2)\). Let \(\sigma : \Sigma_2(E_2) \to \Sigma_2(E_2)\) denote the map
\[
\sigma \left( [\cdots ;(t_{-1}, \omega_{-1}); (t_0, \omega_0); (t_1, \omega_1); \cdots] \right) = [\cdots ;(t_{-1}, \omega_{-1}); (t_0, \omega_0); (t_1, \omega_1); \cdots].
\]

Then one proves in the standard way that \(\sigma\) is a homeomorphism.

**Lemma 4** The homeomorphism \(\sigma : \Sigma_2(E_2) \to \Sigma_2(E_2)\) is transitive and its set of periodic points is dense.

**Proof** The proof is just the compilation of the well known proofs of the similar statements for the classic Smale horseshoe and the Smale solenoid [13, 11, 9], therefore we omit it. □

**Theorem 1** The restriction \(f|_\Omega : \Omega \to \Omega\) of the diffeomorphism \(f\) to \(\Omega\) is conjugate to \(\sigma : \Sigma_2(E_2) \to \Sigma_2(E_2)\).

**Proof** Denote by \(\vartheta : \Omega \to \Sigma_2(E_2)\) the map which assigns to a point \(P \in \Omega\) its code \(\hat{P} \in \Sigma_2(E_2)\). Since the rectangles of any fixed degree are pairwise disjoint, the map \(\vartheta\) is well defined and it is single-valued, i.e. if \(P_1 \neq P_2\) then \(\vartheta(P_1) \neq \vartheta(P_2)\). It is clear that the rectangles of the fixed degree continuously depend (in the Hausdorff topology in the space of compact sets) on the parameter \(t\) of the square \(D_t\), therefore \(\vartheta\) is continuous. Since the space \(\Sigma_2(E_2)\) is compact, \(\vartheta\) is a homeomorphism. It is immediate from the definition of the coding that the diagram
\[
\begin{array}{ccc}
\Omega & \xrightarrow{f|_\Omega} & \Omega \\
\downarrow \vartheta & & \downarrow \vartheta \\
\Sigma_2(E_2) & \xrightarrow{\sigma} & \Sigma_2(E_2)
\end{array}
\]
is commutative. This means that the maps \(f|_\Omega\) and \(\sigma\) are conjugate. □

**Corollary 1** The homeomorphism \(f|_\Omega : \Omega \to \Omega\) is transitive and its set of periodic points is dense.

Consider on \(S^1 \times B^2\) a magnetic field \(\vec{B}\) of unit vectors tangent to the curves \(S^1 \times \{z\}\), \(z \in B^2\). One assumes the curves \(S^1 \times \{z\}\) to be oriented in the direction of the parameter increase. It is clear that \(\vec{B}\) can be extended to the unit vector field (therefore, divergence-free) of the entire sphere \(S^3\). The following result shows that the energy of the magnetic field \(\vec{B}\) grows exponentially with the exponent \(\mu > 0\).

**Theorem 2** The diffeomorphism \(f : S^3 \to S^3\) is fast nondissipative kinematic dynamo with respect to the magnetic field \(\vec{B}\).

**Proof** Since \(f\) stretches the length of the curves \(S^1 \times \{z\}\) two-fold, \(f\) transforms the field \(\vec{B}\) to the field \(f_s(\vec{B})\) with the following property: there is a constant \(\lambda > 1\) such that the vectors of \(f_s(\vec{B})\) are longer by at least factor \(\lambda\) then those of the field \(\vec{B}\). The same holds for the lengths of the vectors of the field \(f_{s+1}^1(\vec{B})\) with respect to the filed \(f_{s+1}^1(\vec{B})\). If we ignore the energy dissipation then the energy of the vector field \(f_n^s(\vec{B})\) grows exponentially with the exponent \(\ln \lambda\). Notice that this result also follows from Lemma 3 and the more general result of the paper [8], in which it is shown that a typical magnetic field is a nondissipative fast dynamo if and only if the diffeomorphism \(f\) has a nonzero topological entropy. □
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References